Computational aspects in creep rupture modelling of polypropylene using an energy failure criterion in conjunction with a mechanical model

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The creep rupture time of polypropylene (PP) was modelled successfully using the Reiner–Weissenberg energy criterion with a mechanical model. The computational aspects were discussed with respect to the software routine used, the equation relating time to rupture and stress, and the various problems associated with non-linear regression analysis for parameters optimization. The predicted upper stress limit for PP homopolymer and copolymer was 30 and 24 MPa, respectively. The predicted lower stress limit was 10 MPa and was independent of polymer type. Short term results can only be used to predict the stress limits if reasonable bounds were imposed on all the parameters. The above approach also predicted reasonable values for the resilience of polypropylene.

(Keywords: creep rupture; polypropylene; Reiner–Weissenberg; non-linear regression; parameter optimization; mechanical model; stress limits; resilience)

INTRODUCTION

Previously an approach which was used to model accurately the creep rupture time of a high density polyethylene was reported[1]. The model predicted the upper stress limit where the specimen ruptures immediately on application of load, and the lower stress limit where, although the specimen creeps it sustains the load indefinitely. Such an approach also modelled accurately the creep rupture time of wood–polymer composites[2].

The creep rupture model is based on Reiner–Weissenberg energy failure criterion[3] in conjunction with a simple three element mechanical model (*Figure 1*) having a linear elastic (Hookean) spring with modulus E_e in series with a parallel arrangement of a rate-activated (Eyring fluid) dashpot and another Hookean spring with modulus E_a. This parallel arrangement gives anelasticity to the model. The Reiner–Weissenberg theory assumes that failure depends upon a maximum value of the intrinsic free energy which can be stored elastically in the volume of a material. Failure will occur when the conserved work (W_c) in the elastic elements reaches a critical value R which is the resilience of the material. From thermodynamics, the conserved work is equal to the difference between the input (W) and the dissipated (D) work. The sum of their difference with respect to time can be expressed as $\dot{W}_c + \dot{D} - \dot{W} = 0$; hence, the Reiner–Weissenberg failure criterion can be stated as

$$W_c = \int_0^{t_f} (\dot{W} - \dot{D})\, dt = R$$

where t_f is the time to failure. This equation shows that failure depends upon the loading history. The first verification of the Reiner–Weissenberg theory of strength was performed by Foux and Brüller[4] on the limit of linear viscoelastic behaviour of perspex and epoxy resin. They found that the resilience values of the two materials were independent of strain rates thus confirming the above theory. For prediction of crazes or fracture a modified version relating only to the energy associated with viscous flow process in polymers was found to be most appropriate[5].

Applying the Reiner–Weissenberg failure criterion to simple mechanical models such as the Kelvin and Maxwell models under constant load gives the following interesting conclusions[3]. A Kelvin solid fails when the strain reaches a critical limiting value and a Maxwell solid fails when the stress reaches a critical limiting value.

When applied to the model as shown in *Figure 1* under creep conditions different critical stress limits can be obtained. These limits relate to the upper and lower stress limits, as derived in previous publications[1,2]. It is important to note that this model does not exhibit permanent plastic deformation. However, this model can still be used for creep rupture prediction on the assumption that failure is defined as the onset of permanent plastic deformation.

To date no related published work, apart from those mentioned, on application of this approach to other polymers has been reported. Communications with those who tried, indicated that the primary problem lies in the computational aspects in the equation which was used to model the creep rupture time. The present paper addresses this point by first examining the model equation critically and establishes a subroutine program based on commercial software packages that could help

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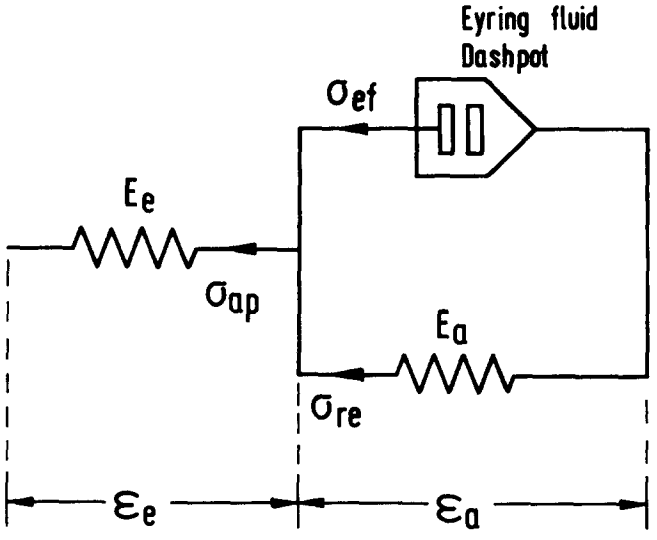

Figure 1 A simple three element mechanical model

researchers in similar fields to obtain quick, easy and consistent results. It extends the above approach to another polymer namely polypropylene, investigating the significance of the lower stress limit which is the most important for engineering design and the possibility of using short term creep rupture results to predict the stress limits. Computed values for the resilience of polypropylene are also discussed.

EQUATION FOR MODELLING CREEP RUPTURE

Basically the equation for creep rupture modelling is expressed as:

$$t_r=[1/(E_aBK)][\ln(\tanh(B\sigma_{ap}/2)\tanh(BH/2))] \quad (1)$$

where

$$H=\sigma_{ap}-[2E_a(R-\sigma_{ap}^2/E_e)]^{1/2}$$

σ_{ap} = applied stress; E_e = elastic modulus; E_a = anelastic modulus; B = parameter related to activation volume; K = parameter related to activation energy, and R = resilience. For the sake of continuity and clarity the derivation of equation (1) is shown in Appendix A.

Equation (1) has two important consequences. It predicts the upper stress (SX) at which the specimen ruptures immediately on application of load. It can be seen that t_r approaches zero if $\sigma_{ap}=(E_eR)^{1/2}$. Consequently one can write

$$SX=(E_eR)^{1/2} \quad (2)$$

It also predicts the lower stress limit (SN) at which the specimen sustains the load indefinitely. It can be seen that t_r approaches infinity if $\sigma_{ap}=[R/(1/E_e+1/2E_a)]^{1/2}$. In terms of SN, one can write

$$SN=[R/(1/E_e+1/2E_a)]^{1/2} \quad (3)$$

It is noteworthy to mention that SX is only dependent on the elastic modulus, E_e, and the resilience, R which is the critical sum of the elastic stored energy. SN is dependent on E_e, R, and the anelastic modulus, E_a which is related to the elastic spring that brings about the time-dependent component of the material. Interestingly, both these parameters are not related to the activation volume parameter, B, and the activation parameter, K. B and K play a major role only in modelling the creep rupture time. Now it appears that one needs five parameters: E_e, E_a, B, K and R, to describe accurately equation (1). However this can be reduced. Using equations (2) and (3), H can be written in terms of SN and SX as:

$$H=\sigma_{ap}-SN[(SX^2-\sigma_{ap}^2)/(SX^2-SN^2)]^{1/2}$$

Thus equation (1) becomes

$$t_r=[1/(CB)\ln[\tanh(B\sigma_{ap}/2)/\tanh(BH/2)] \quad (4)$$

where $C=E_aK$.

Equation (4) suggests only four parameters for modelling t_r. The four parameters are SX, SN, B and C. The reduction of one parameter greatly reduces computing time during parameter optimization.

Using equations (2) and (3) also yields

$$E_e/E_a=2(SX^2-SN^2)/SN^2 \quad (5)$$

which gives the ratio of the elastic and anelastic modulus as a function of the upper and lower stress limits. Equation (5) dictates that, at the present moment, only the ratio E_e/E_a can be obtained. Individual values of E_e and E_a reported earlier[1,2] are therefore subject to speculation. With regard to the computation of the four parameters used in this model some difficulty is encountered. Primarily, this is due to the non-linear relationship as expressed in equation (4), between creep rupture time and the four parameters. As a result the computer optimization program is highly dependent on the initial estimates of the parameters. This sometimes leads to numerical values of the parameters that do not have physical meaning. Moreover, the choice of a computer software program to perform the non-linear regression analysis is also found to be critical. The wrong choice of program and improper programming algorithms can lead to non-convergence, excessive computing time and poor parameter estimates.

Statistical computation

A number of commercial software packages are available to carry out non-linear regression analysis, for example the International Mathematical and Statistical Library (IMSL) ZXSSQ[6] and the Statistical Analysis System (SAS) NLIN[7]. Both these packages can run on an IBM mainframe. The IMSL ZXSSQ routine is a finite difference, Levenberg–Marquardt routine for solving non-linear least squares problems[8–11]. This routine is based on a modification of the Levenberg–Marquardt algorithm which eliminates the need for explicit derivatives. On the other hand the SAS NLIN implements iterative methods that attempt to find the least squares estimates of the parameters for non-linear models. This routine has four options available namely the Gauss–Newton[8], Marquardt[11], Steepest Descent[12] and Secant[13] method. The advantage of SAS NLIN is that it performs an initial grid search of the parameters and uses the best set for the initial trial estimates. This eliminates a significant amount of time in obtaining initial estimates of the parameters by trial and error method. Because of this, the SAS NLIN routine is preferred for use in the above approach for creep rupture modelling. It is also found that for most consistent results, the Marquardt[11] option in SAS NLIN is the best. The Marquardt method

is a compromise between the Gauss–Newton and Steepest Descent method. It has been proven to be very reliable in practice and in several cases faster than other methods[9]. It is equivalent to performing a series of ridge regressions and is most useful when the parameter estimates are highly correlated[14].

EXPERIMENT

Creep rupture testing

The details for creep rupture testing have been described previously[1]. All tests were performed under constant temperature of 20 ± 1°C and relative humidity of 65 ± 5%. Dead weights were hung on to specimens in a uniaxial direction with the help of a hydraulic jack. The creep rupture time was automatically recorded by a clock timer. The duration of the tests varied from a few seconds to more than three years. If the specimen did not break, the time taken for it to reach the limit of the test rig (about 300% extension) was taken as the time to failure. The long time creep rupture tests were necessary in order to confirm the lower stress limit predicted by equation (4). All applied stress was calculated by dividing the load by the original cross sectional area. The applied stress ranges from 10.06 to 24.06 MPa.

Material and sample preparation

Polypropylene (PP) was used. This material was supplied by Idemitsu, Japan, code labelled Idemitsu PPJ-452H. It came in white pellet form with a melt flow index of 3.0 g/10 min (at 230°C with a load of 216 kg). The PP was injection moulded in a simple laboratory size injection moulder (Unimoulder, UK) having a ram diameter of 25 mm and 30 g per shoot, to a shape conforming to ASTM D638, Type I. Edge gating with a fan gate configuration was used. The injection pressure was set at 1.03 MPa and the injection temperature at 180°C. The single cavity mould was cooled by running water at 27°C. The injected PP specimen was left in the mould for 45 s before it was taken out. The average cooling rate was 3°C/s.

Before creep rupture testing, the PP specimens were machined to ASTM D638, Type II. This was done to reduce the weight that was required for creep rupture testing because the cross-sectional area at the gauge length of the Type II configuration is much smaller than Type I. All machining marks were removed by using a fine abrasive paper (grade 00). A total of 24 specimens were prepared for creep rupture testing.

Computational aspects

Because the model equation (4) involves non-linear parameter optimization involving many different trials and iterations, all computations were carried out on an IBM host computer (IBM 3801) with SAS NLIN as the software package for non-linear parameters optimization.

In order to use the SAS NLIN program, the PROC NLIN in SAS has to be invoked. To invoke this routine, the following statements were required:

```
PROC NLIN options
  PARAMETERS (PARMS) parameters=values ...;
  BOUNDS expression ...;
    other programming statements;
  MODEL dependent=expression;
  DER parameter=expression;
  OUTPUT out=SASdataset keyword=variable;
```

Briefly, the PROC NLIN options determine the data set to be used, the method for the iteration and other values to control the iteration. The PARAMETERS statement identifies the parameters to be estimated and their initial estimates. If a range is given for the parameters, a grid search is performed such that the combination of the parameters with the lowest error sum of squares (ESSQ) is chosen as the initial estimates. This is important as it reduces computing time when using a trial-and-error method to estimate the starting initial values. The BOUNDS expression retains the parameter estimates within specified bounds. This expression is important in order to give parameter estimations that are meaningful. The MODEL statement defines the prediction equation by declaring the dependent variables and defining an expression that evaluates the predicted values. The DER statements represent the partial derivatives of the MODEL statement with respect to the parameters. This is required for the Marquardt method. If these statements are missing, SAS NLIN will use the DUD[11] method to solve the problem. The OUTPUT statement specifies an output data set to contain statistics calculated for each observation.

RESULTS AND DISCUSSIONS

Computational aspects

A sample of the SAS NLIN program is shown in Appendix B using polypropylene creep rupture test data. The program uses Y5 to denote $\ln t_r$, and X5 to denote the applied stress. 'If....Else Do;' statements were used to ensure that the program does not run into problems due to arithmetic division by zero or choosing non-meaningful negative variables during the iterative process. The importance of the BOUNDS expression (3rd paragraph after the INPUT data set in Appendix B) must be emphasized. *Table 1* shows the results if no upper bound was placed on SX. It can be seen that the values of the ESSQ keep on improving even until $SX = 5 \times 10^{14}$ MPa, a value of no practical meaning. An estimate of the upper bound for SX can be taken to be 1 or 2 MPa above that of the instantaneous (1 s) fracture stress measured at the intersection of the extrapolated linear regressed line to the X-axis representing $\ln t_r = 0$, that is, the stress that produces a 1 s rupture time (*Figure 2*). The lower bound for SX is usually set between 5 to 10 MPa below the upper bound. This is to take into account that short time fracture stress measured experimentally is not accurate because rapid loading at high stresses may give rise to non-uniform temperature distribution along the gauge length of the specimen. The

Table 1 Effect of having no upper bound on SX in parameters optimization of equation (6)

SX (MPa)	SN (MPa)	B (1/MPa)	C (MPa/s)	ESSQ
24	10	0.54	8.90E-7	19.43
25	10	0.56	7.64E-7	18.48
30	10	0.70	3.45E-7	13.32
35	10	0.73	3.26E-7	13.24
40	10	0.76	2.96E-7	12.72
45	10	0.78	2.87E-7	12.33
50	10	0.80	2.76E-7	12.13
–	–	–	–	–
5 E+14	10	0.83	3.09E-7	7.44

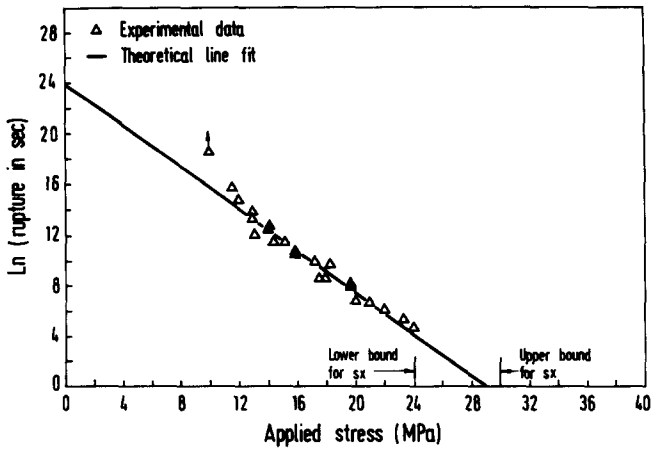

Figure 2 Linear regression analysis of PPJ-452H. Linear regression equation is $\ln t_r = 24.07 - 0.83\sigma_{ap}$

upper and lower bounds for the other parameters (SN, B and C) are difficult to establish, and from experience, it is sufficient just to have a bound statement requiring these parameters to be greater than zero.

Convergence can be reasonably expected only with fully identified parameters, adequate data, proper bound statements, and good initial starting values sufficiently close to the solution estimates. Jonathan Bard[12] writes: 'The reader should realize that the state of the art of non-linear optimization is such that one cannot as yet write a computer program that will produce the correct answer to every parameter estimation problem in a single computer run. All too often, the first run produces unacceptable results. By studying these results one can perhaps obtain better starting guesses'. This quotation underscores the importance of getting reasonable initial estimates. The SAS NLIN program allows the user to specify a range for each parameter to be estimated and also the step change required within the range for performing the grid search for the initial estimates. Small step change is avoided since the grid search can take a long time. This provision of a grid search has been found to be very useful if no reasonable initial estimates are known for the first trial values.

The above non-linear optimization procedure is also sensitive to the type of model equation used in the software routine. It has been found that equation (4) when used as it stands runs into computational problems related to exponential overflow especially in the estimate for the residuals which is related to the error sum of squares obtained from the difference of the experimental values and the predicted values. To avoid such problems equation (4) was written as

$$\ln t_r = \ln \{[1/CB)] \ln [\tanh (B\sigma_{ap}/2)/\tanh (BH/2)]\} \quad (6)$$

This equation (6) was used throughout the creep rupture time modelling and has been found to be very successful for other creep rupture data of other polymers (to be published later).

Significance of the lower stress limit

Figure 3 shows a plot of ln(time to rupture in seconds) *versus* applied stress using all the creep rupture data. All the specimens show extensive cold drawing before ultimate failure. The continuous curve line is the fit predicted by equation (6). The actual experimental data are in triangles. It can be seen that the fit by equation (6), with the proper bound statements as shown in Appendix B, is a good one. Equation (6) predicted $SX = 30$ MPa, $SN = 10$ MPa, $B = 0.65$ MPa^{-1} and $C = 5.70 \times 10^{-7}$ (MPa/s) with ESSQ = 8.78. A point to note is that at a stress of 10.06 MPa, no rupture was observed even after 3 years of creep rupture testing (indicated by an arrow head in *Figure 3*). It is believed that $SN = 10$ MPa is a reasonable lower stress limit for polypropylene. However, in order to substantiate this point creep rupture data of polypropylene (homo and copolymer) from Gotham[15], were analysed using equation (6). These polypropylene samples were injection moulded and creep rupture tested also at 20°C. Their creep rupture data were numerically obtained by using a digital plotter and admittedly, some errors were therefore inherent but considered to be small because of the log function used for the time to rupture. Their results in term of ln(time to rupture) *versus* applied stress are shown in *Figures 4* and *5*. *Table 2* shows their computed results. Although there is no indication from the supplier for the polypropylene (Idemitsu PPJ-452H) under study, whether it is a homo- or copolymer, it is interesting to note, regardless of this information, that the lower stress limit of 10 MPa seems to be a realistic value to adopt for polypropylene. Furthermore the British Standards Institute (Code of practice 312) has designated a safe

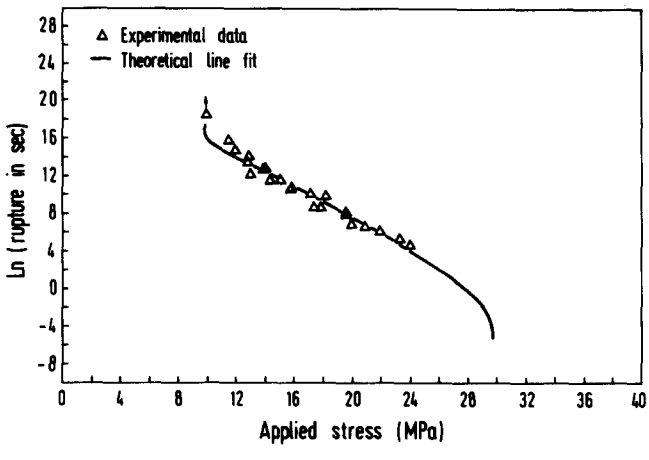

Figure 3 Modelling the creep rupture time of PPJ-452H using equation (6): SX = 30 MPa, SN = 10 MPa

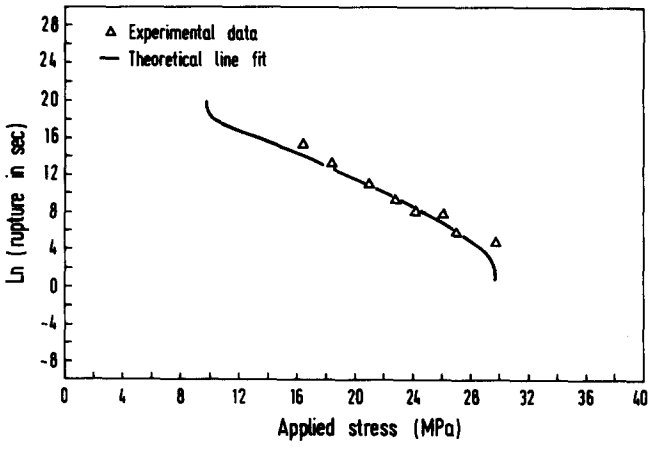

Figure 4 Modelling the creep rupture time of PP homopolymer[15] using equation (6): SX = 30 MPa, SN = 10 MPa

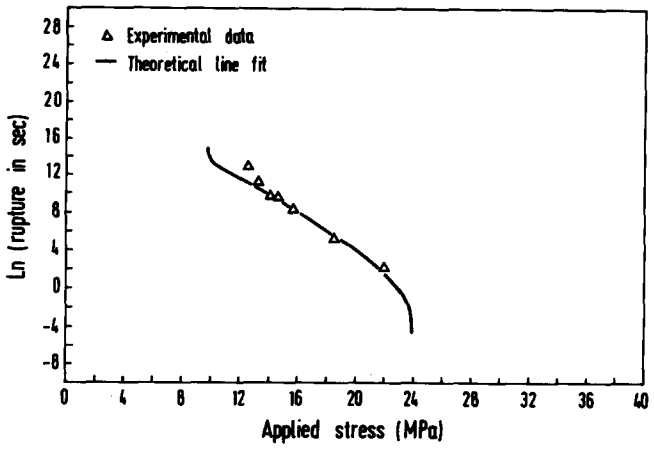

Figure 5 Modelling the creep rupture time of PP copolymer[15] using equation (6): SX = 24 MPa, SN = 10 MPa

Table 2 Computed results for polypropylene (PP)

Material	SX (MPa)	SN (MPa)	B (1/MPa)	C (MPa/s)	ESSQ
PPJ-452H	30	10	0.65	5.70E-7	8.78
PP-copolymer[15]	24	10	0.66	4.45E-6	0.92
PP-homopolymer[15]	30	10	0.54	4.47E-8	1.30

design stress of 5 MPa for long term usage of polypropylene, and assuming a reasonable safety factor of 2, it can be seen that the computed lower stress limit as predicted by equation (6) can be considered to be a good estimate for the lower stress limit before the application of a safety factor. It is noteworthy to mention that the fatigue endurance limit as reported by Riddell[16] for polypropylene was also about 10 MPa. This coincidence is important as it suggests that the lower stress limit can also be taken to be the endurance limit under fatigue conditions. This is not illogical as one envisaged fatigue failure to be associated with crack initiation leading to crack propagation and ultimately to fracture, a process involving chain sliding and breaking, which according to the present creep rupture model is related to the chains having been stretched to such a state where the elastic stored energy reaches a critical value, R. However, it will be interesting to find out whether the lower stress limit does correspond to the endurance limit for other polymers, a topic for future work.

Predicting SN using short term results

In an effort to see whether short term creep rupture results can be used to predict SN, the creep rupture data for 26.4, 6.5, 1.09 and 0.23 days were used. Bound statements were imposed as mentioned earlier (see Appendix B). However much smaller steps were used in the grid search. The computed results are shown in *Table 3*. Here it can be seen that SN varies from about 5 to 8 MPa. It has to be mentioned that a long computing time was used for the extensive grid search for obtaining the results in *Table 3*. This was necessary in order to obtain the best initial estimate of the four parameters. Initially the above exercise was intended to investigate whether there is a trend towards the minimum duration of creep rupture testing to give a reasonable SN value close to 10 MPa. It is obvious from *Table 3* that such a trend is not there. One would expect for longer time, the value of SN should be closer to 10 but it turned out to be the reverse. However, it is clear from the values of SX, the magnitude is highly predictable. This is because the upper and lower bounds for SX had been imposed. One can therefore logically conclude that unless similar bounds can be imposed on the other parameters, using short term creep rupture results to predict SN is highly speculative. Nonetheless, *Table 3* suggests that if such bounds can be determined, one only need creep rupture results over perhaps less than a day to give a reasonably close value of the true SN value.

Resilience

From equation (2), the value of the resilience (R) can be computed if one assumes a certain value for E_e for PP. Taking E_e to be that quoted in refs 17 and 18, as 41 GPa and using equation (2), $R = SX^2/E_e$, hence for PPJ-452H and PP-homopolymer[15] the R value is 0.02195 MJ/m^3 and for PP-copolymer the value is 0.01404 MJ/m^3. No published literature on PP can be cited for comparison. However the resilience of perspex[4], based on the Reiner–Weissenberg criterion, has been calculated to be 0.084 MJ/m^3. Previous values for a high density polyethylene[1] was in the order of 0.003 MJ/m^3. According to Reiner and Weissenberg[19], the resilience is independent of the strain rate and, hence, can be considered as a material property. This has been found true from the experimental results of Foux and Brüller[4] on the resilience of perspex and epoxy.

CONCLUSION

Modelling the creep rupture time using a three element mechanical model in conjunction with the Reiner–Weissenberg energy failure criterion was found to apply for polypropylene. The non-linear relationship between log(time to rupture) and applied stress dictates that the parameters used in the model equation need to be optimized by non-linear regression analysis. The SAS NLIN regression analysis program using the Marquardt's method of convergence, was found to be most suitable. Meaningful estimates of the parameters, which have been reduced to four from five, as in a previous publication[1], can best be made if proper bounds are imposed on them. For engineering design purposes, the lower stress limit is the most important and this limit was found to be equal to 10 MPa regardless of whether the polymer is a homopolymer or copolymer. This lower stress limit cannot be predicted by using short term creep rupture test results unless reasonable bounds were imposed on all the parameters. The upper stress limit was found to be dependent on the type of polymer. For the homopolymer it was 30 MPa and the copolymer it was 24 MPa.

The ratio of the elastic to anelastic modulus is only dependent on the upper and lower stress limits. The

Table 3 Parameters estimation of PPJ-452H using short term results

Time (days)	SX (MPa)	SN (MPa)	B (1/MPa)	C (MPa/s)	ESSQ
26.4	30	5.07	0.71	1.56E-9	6.21
6.5	30	4.10	0.69	1.28E-9	4.33
1.09	30	8.01	0.60	4.33E-7	2.73
0.23	30	8.03	0.57	6.60E-7	2.52

creep rupture model can also be used to predict reasonable values for the resilience of PP. The resilience for PP-homopolymer was 0.02195 MJ/m^3 and for PP-copolymer was 0.01404 MJ/m^3.

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APPENDIX A

Creep rupture model

The mathematical formulation of the Reiner–Weissenberg energy criterion of failure can be described, for the model shown in *Figure 1*, as:

$$\int_0^{\varepsilon_e^*} \sigma_{ap}\, d\varepsilon_e + \int_0^{\varepsilon_a^*} \sigma_{re}\, d\varepsilon_a = R \tag{7}$$

where ε_e^* = elastic strain at rupture; ε_a^* = anelastic strain at rupture; σ_{ap} = applied stress; σ_{re} = recovery stress and R = critical sum of the elastic stored energy (often referred to as the resilience of the material which is a material property related to the intrinsic free energy that can be stored elastically in the volume element of the material).

The first integral in equation (7) relates to the elastic stored energy for the spring element E_e and the second integral relates to the elastic stored energy for the spring E_a.

Under creep conditions, the applied stress is constant. Hence, by using $\varepsilon_e^* = \sigma_{ap}/E_e$, the first integral can be evaluated as σ_{ap}^2/E_e. For the second integral, by using the relationship $\sigma_{re} = \varepsilon_a E_e$ where ε_a and E_a are the anelastic strain and modulus respectively, it can be evaluated as $(\varepsilon_a^*)^2 E_a/2$.

Equation (7) can be rearranged to make ε_a^* as the dependent variable as

$$\varepsilon_a^* = [(R - \sigma_{ap}^2/E_e)(2/E_a)]^{1/2} \tag{8}$$

Now, the anelastic strain rate ($\dot{\varepsilon}_a$) can be expressed in terms of the effective stress (σ_{ef}) using the hyperbolic sinh function, which is used to describe an Eyring's fluid, as

$$\dot{\varepsilon}_a = K \sinh (B\sigma_{ef}) \tag{9}$$

where K is a function of the activation energy which is a constant at fixed temperature, and B is a function of the activation volume (V) and is equal to $V/2kT$ (k and T represent the Boltzmann's constant and the absolute temperature, respectively).

However, the anelastic strain rate can also be related to the recovery stress and the anelastic modulus as

$$\dot{\varepsilon}_a = \dot{\sigma}_{re}/E_a \tag{10}$$

The applied stress is the sum of the effective and recovery stress, i.e. $\sigma_{ap} = \sigma_{re} + \sigma_{ef}$. The time derivative of this equation can be written as

$$\dot{\sigma}_{ap} = \dot{\sigma}_{re} + \dot{\sigma}_{ef} \tag{11}$$

For creep conditions, $\dot{\sigma}_{ap} = 0$, since the applied stress is constant. Hence from equation (11), $\dot{\sigma}_{re} = -\dot{\sigma}_{ef}$. Putting this in equation (10) and equating the resultant formula to equation (9), the following expression can be derived:

$$\int_{\sigma_{ap}}^{\sigma_{ef}} d\sigma_{ef}/\sinh (B\sigma_{ef}) = -\int_0^t E_a K\, dt \tag{12}$$

where t = time. The limits in the left hand integral of equation (12) are derived from the fact that at time equal to zero, the applied stress is acting only on the Eyring dashpot because the spring is not extended at this time.

After evaluation of the integrals and rearrangement to make σ_{ef} the dependent variable, equation (12) becomes

$$\sigma_{ef} = \tfrac{2}{B} \tanh^{-1} [\tanh (B\sigma_{ap}/2) \exp (-E_a BKt)] \tag{13}$$

Because σ_{re} can be expressed as $(\sigma_{ap} - \sigma_{ef})$, the anelastic strain ε_a can be written as

$$\varepsilon_a = (\sigma_{ap} - \sigma_{ef})/E_a \tag{14}$$

Putting equation (13) in equation (14) gives

$$\varepsilon_a = (\sigma_{ap}/E_a) - (2/BE_a) \tanh^{-1} [\tanh (B\sigma_{ap}/2) \exp (-E_a BKt)] \tag{15}$$

Now equation (15) can be used to eliminate ε_a^* in equation (8) by noting that when $t = t_r$, the rupture time, $\varepsilon_a = \varepsilon_a^*$.

Hence by making t_r the dependent variable,

$$t_r = (1/E_a BK) \ln [\tanh (B\sigma_{ap}/2)/\tanh (BH/2)] \tag{16}$$

where $H = \sigma_{ap} - [2E_a(R - \sigma_{ap}^2/E_e)]^{1/2}$.

This then describes the creep rupture modelling equation relating the rupture time, t_r, to the applied stress, σ_{ap}, using the parameters E_e, E_a, B, R and K.

APPENDIX B

SAS NLIN subroutine for creep rupture modelling

```
**********************************************
THIS PROGRAM IS FOR CREEP RUPTURE
MODELING USING THE SAS NLIN
(NON-LINEAR REGRESSION) SUBROUTINE
**********************************************
```

```
CMS FI 01 DISK TEOHSH 1507S01 (RECFM VBS
LRECL 364 BLOCK 368; *GOPTION OVP
DEVICE=CAL1051 UNIT=01 HSIZE=20
VSIZE=20; GOPTION DEVICE=IBM32793;

DATA A;

INPUT                Y5          X5;
CARDS;             18.35       10.06
                   15.62       11.67
                   14.64       12.03
                   13.88       13.01
                   13.24       12.91
                   12.74       14.12
                   12.55       14.03
                   12.02       13.08
                   11.45       14.44
                   11.36       15.19
                   10.58       16.00
                   10.56       15.93
                   10.44       15.88
                    9.88       17.24
                    9.64       18.36
                    8.58       17.51
                    8.60       18.00
                    8.05       19.68
                    7.80       19.68
                    6.72       20.00
                    6.49       21.01
                    5.99       21.98
                    5.13       23.36
                    4.50       24.06

;
PROC NLIN DATA=A
METHOD=MARQUARDT BEST=10
  EFORMAT MAXITER=2000;

PARMS SX=24  TO 30 BY 1  SN=0  TO 10  BY 1
  B=0.01 TO 1 BY 0.1  C=1E-9 TO 1E-6 BY 1E-8;

BOUNDS 24<SX<30, SN>0, B>0, C>0;

    S0=SX**2-X5**2;
    S1=SX**2-SN**2;
  IF S1<=0 THEN RETURN;
  ELSE DO;
    Z=S0/S1;
  IF Z<=0 THEN RETURN;
  ELSE DO;
    Y=SQRT(Z);
    H=X5-SN*Y;
  IF H<=0 THEN RETURN;
  ELSE DO;
    X=B*X5/2;
    W=B*H/2;
    V=TANH(X);
    U=TANH(W);
    S=LOG(V/U);
  IF S<=0 THEN RETURN;
  ELSE DO;
    T=SINH(X*2);
    Q=SINH(W*2);
    A=S/(C*B);
  IF A<=1 THEN RETURN;
  ELSE DO;

MODEL Y5=LOG(A);

DER.SX=B*SX*SN*Y*(1/S0-1/S1)/(S*Q);
DER.SN=Y*(1-SN**2)*B/(S*Q);
DER.B=(-1)/B+(X5/T-H/Q)/S;
DER.C=(-1)/C;
  END;
  END;
  END;
  END;
  END;

OUTPUT OUT=N P=YTHEO R=YRESID
PARMS=SX SN B C ESS=ESSQ;

TITLE POLYPROPYLENE (INJECTION
MOLDED);

PROC PRINT;

PROC PLOT DATA=N;

PLOT Y5*X5="O" YTHEO*X5='*'/OVERLAY

VAXIS=0 TO 20 BY 1

HAXIS=0 TO 90 BY 5;
```